AN OPTIMAL EXTENSION OF PERELMAN'S COMPARISON THEOREM FOR QUADRANGLES AND ITS APPLICATIONS

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Abstract: In this paper we discuss an extension of Perelman's comparison for quadrangles. Among applications of this new comparison theorem, we study the equidistance evolution of hypersurfaces in Alexandrov spaces with non-negative curvature. We show that, in certain cases, the equidistance evolution of hypersurfaces become totally convex relative to a bigger sub-domain. An optimal extension of 2nd variational formula for geodesics by Petrunin will be derived for the case of non-negative curvature.

In addition, we also introduced the generalized second fundament forms for subsets in Alexandrov spaces. Using this new notion, we will propose an approach to study two open problems in Alexandrov geometry.

Keywords: Alexandrov spaces, quasi-geodesics, comparison theorem for quadrangles

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§0. Introduction

In this paper, we derive a sharp comparison theorem for quadrangles in complete metric spaces with non-negative curvature. An earlier version of such a comparison

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for quadrangles was discovered by Perelman with an asymptotical estimate (cf. §6 of [Per91]). For smooth Riemannian manifolds with non-negative sectional curvature, such sharp comparison theorem was implicitly stated in an important paper [CG72] of Cheeger-Gromoll. We will extend Cheeger-Gromoll's approach to singular spaces of non-negative curvature.

Among applications of our new sharp comparison for quadrangles, we will derive a sharp version of the 2nd variational formula of lengths in curved singular spaces, which improves the earlier work of Petrunin (cf. [Petr98]). Other applications of our new sharp comparison is to study the changes of Hessians of distance functions in non-negatively curved spaces. This application would provide a curved version of moving half-space method in Alexandrov spaces with non-negative curvature, which we address in a separate paper (cf. [CDM07]).

In §6 of an important preprint [Per91], Perelman pointed out an asymptotic estimate for a class of quadrangles. Perelman's work was completed before the notion of quasi-geodesic segments were introduced. We will recall the definition of quasi-geodesics in §1 below. In what follows, we always let $\sigma_{pq} : [0, \ell] \to M^n$ be a quasi-geodesic segment from p to q of unit speed.

Definition 0.1. *Let $\triangle_1$ be a quasi-geodesic triangle with sides $\{\sigma_{qp}, \sigma_{q\hat{p}}, \sigma_{p\hat{p}}\}$ and $\triangle_2$ be a quasi-geodesic triangle with sides $\{\sigma_{q\hat{p}}, \sigma_{q\hat{q}}, \sigma_{\hat{p}\hat{q}}\}$. If the two triangles have a common side $\sigma_{q\hat{p}}$ and if*

$$\angle_q(\sigma'_{qp}(0), \sigma'_{q\hat{p}}(0)) + \angle_q(\sigma'_{q\hat{p}}(0), \sigma'_{q\hat{q}}(0)) = \angle_q(\sigma'_{qp}(0), \sigma'_{q\hat{q}}(0)) \tag{0.1}$$

holds, then we say that the two quasi-geodesic triangles $\triangle_1$ and $\triangle_2$ are co-planar at q.

Theorem 0.2. *(Extended Perelman's comparison for quadrangles) Let M^n be a complete Alexandrov space of curv ≥ 0, $\triangle_1$ and $\triangle_2$ be two quasi-geodesic triangles co-planar at q as above. Suppose that $\{\sigma_{q\hat{q}}, \sigma_{p\hat{p}}\}$ are the only two possible quasi-geodesics and the remaining quasi-geodesic segments $\{\sigma_{\hat{q}\hat{p}}, \sigma_{q\hat{p}}, \sigma_{pq}\}$ are*

length-minimizing geodesic segments with

$$\angle_q(\sigma'_{qp}(0), \sigma'_{q\hat{p}}(0)) + \angle_q(\sigma'_{q\hat{p}}(0), \sigma'_{q\hat{q}}(0)) = \angle_q(\sigma'_{qp}(0), \sigma'_{q\hat{q}}(0)) = \frac{\pi}{2}. \tag{0.2}$$

Then

(1) The function $s \to d(\sigma_{p\hat{p}}(s), \sigma_{q\hat{q}}(\mathbb{R}))$ *is concave at* $s = 0$*:*

$$\frac{d[f(\sigma_{p\hat{p}}(s))]}{ds}(0) \leq \cos\theta, \tag{0.3}$$

where $f(x) = d(x, \sigma_{q\hat{q}}(\mathbb{R}))$*.*

(2) If, in addition, $\sigma_{p\hat{p}}$ *is a length-minimizing geodesic segment, then we have the following sharp estimates*

$$d(\hat{p}, \sigma_{q\hat{q}}(\mathbb{R})) \leq d(p, q) - s\cos\theta \tag{0.4}$$

where $s = d(p, \hat{p})$ *and* $\theta = \angle_p(\sigma'_{p\hat{p}}(0), \sigma'_{pq}(0))$*. Moreover, the equality holds in (0.4) if and only if there is* $\tilde{q} \in \sigma_{q\hat{q}}(\mathbb{R})$ *such that four points* $\{p, \hat{p}, \tilde{q}, q\}$ *span a totally geodesic flat trapezoid.*

For the special case of smooth Riemannian manifolds with non-negative sectional curvature, the comparison theorem for quadrangles above was due to Cheeger-Gromoll, see the proof of Theorem 1.10 of [CG72].

Some independent work were carried out by Alexander-Bishop [AB08] via a different method. Among other things, Alexander and Bishop used the ratios of arc/chord and base-angle/chord to measure the convexity of hypersurfaces, which are very interesting. There are some overlap between our Corollary 2.4 and their results. On one hand, our hypothesis of corollary also allows points of zero geodesic curvature for curves in surfaces, e.g., the super-graph of $y = x^4$ is strictly convex in our sense but the boundary has zero geodesic curvature at the origin. Alexander-Bishop's definition of strictly convexity is not applicable to case of super-graph of $y = x^4$. On the other hand, Alexsnder-Bishop's work [AB08] covers not only spaces with curvature $\geq k$ but also spaces with curvature $\leq C$. Their results cover more ambient spaces.

§1. Earlier results on quasi-geodesics, development maps, classical comparisons and stability of Alexandrov spaces

In this section, we review some earlier results for Alexandrov spaces, which will be used in later sections of our paper. Other basic materials for Alexandrov spaces can also be found in Chapter 10 of textbook [BBI01] and [BGP92].

A complete Alexandrov space M^n is often referred to a complete metric space with curvature bounded below. Since Alexandrov and his Russian school of geometry used the geometric triangles to define "the space with $curv \geq -k$", we require that M^n must be a length space. Let $T_x^- M^n$ be the tangent cone of an Alexandrov space M^n at x. With some additional efforts, one can show that the tangent cone $T_x(M)$ must have angular measurement. In fact, the angular measurement is a distance function defined on the unit tangent cone $\Sigma_x(M^n)$ of M^n at x. We now recall some important features of Alexandrov spaces as follows.

Definition 1.1. *(1) A complete metric space M^n is called a length space (or an inner space) if for any pair of points $\{p, q\}$ in M^n, there exists a length-minimizing curve $\sigma : [0, \ell] \to M^n$ from p to q such that the length $L(\sigma)$ of σ is equal to $d(p, q)$, the distance between p and q. Such a length minimizing curve σ is called a geodesic segment of M^n.*

(2) Suppose that (M^n, d) is a complete length space. We say that the space (M^n, d) has $curv \geq 0$, if for any length minimizing geodesic segment $\sigma : [0, \ell] \to M^n$ of unit speed and $p \notin \sigma([0, \ell])$, the function

$$\eta^0_{p,\sigma}(t) = \frac{1}{2}[d(p, \sigma(t))]^2 - \frac{t^2}{2} \tag{1.1}$$

is a concave function, i.e.

$$\eta^0_{p,\sigma}\left(\frac{t_1 + t_2}{2}\right) \geq \frac{1}{2}[\eta^0_{p,\sigma}(t_1) + \eta^0_{p,\sigma}(t_2)], \tag{1.2}$$

for all $\{t_1, t_2\} \subset [0, \ell]$.

Roughly speaking, if the space (M^n, d) has *curv* ≥ 0, then the triangle $\triangle_{p,\sigma}$ spanned by $\{p, \sigma\}$ is more concave towards the vertex p than the corresponding triangle $\triangle^*_{p^*,\sigma^*}$ in the Euclidean space $\mathbb{R}^2$.

Notice that

$$[(t+c)^2]'' = (t^2)''$$

for any constant c. Thus, in Definition 1.1 for the space with *curv* ≥ 0, we use the comparison t^2 instead of $(t+c)^2$ for appropriate c.

In order to define the Alexandrov space with *curv* ≥ -1, for any given pair $\{p, \sigma\}$ with $p \notin \sigma([0, \ell])$, we choose the corresponding pair $\{p^*, \sigma^*\}$ in the hyperbolic plane $\mathbb{H}^2$ of constance curvature -1 more carefully as follows. Let

$$\varphi_{p,\sigma}(t) = d(p, \sigma(t)). \tag{1.3}$$

For any pair $\{p^*, \sigma^*\} \subset \mathbb{H}^2$, we let

$$\varphi^*_{p^*,\sigma^*}(t) = d_{\mathbb{H}^2}(p^*, \sigma^*(t)),$$

where $\sigma^* : [0, +\infty) \to \mathbb{H}^2$ is a geodesic of unit speed. We require that

$$\begin{cases} \varphi_{p,\sigma}(0) = \varphi^*_{p^*,\sigma^*}(0), \\ \frac{d\varphi_{p,\sigma}}{dt}(0) = \frac{d\varphi^*_{p^*,\sigma^*}}{dt}(0). \end{cases} \tag{1.4}$$

Definition 1.2. *Let (M^n, d) be a length space, $\{p, \sigma\}$, $\{p^*, \sigma^*\}$, $\varphi_{p,\sigma}$, φ_{p^*,σ^*} be as above. We say that the space (M^n, d) has curv ≥ -1 if for any geodesic $\sigma : [0, \ell] \to M^n$ of unit speed and $p \notin \sigma([0, \ell])$, the function $f_{-1}(t) = f_{p,\sigma,-1}(t) = -[1 - \cosh(d(p, \sigma(t))]$ satisfies the differential inequality:*

$$f''_{-1} \leq [1 + f_{-1}]. \tag{1.5}$$

Similarly, for $k = 1$, we let $\rho_1(s) = [1 - \cos s]$, $f_{p,\sigma,1}(t) = \rho_1(d(p, \sigma(t))$. If $f_1(t) = f_{p,\sigma,1}(t)$ satisfies the differential inequality:

$$f''_1 \leq [1 - f_1] \tag{1.6}$$

for all length-minimizing geodesic σ, then we say that M^n has curv ≥ 1.

It is well-known that if (M^n, d) has *curv* $\geq k$, then $(M^n, \lambda d)$ has *curv* $\geq \frac{k}{\lambda^2}$, where $\lambda > 0$. By scaling the distance function with a factor $\lambda > 0$, we can define the notion "*curv* $\geq k$" for any real number $k \in \mathbb{R}$.

Burago, Gromov and Perelman [BGP92] derived several important results for Alexandrov spaces. Among other things, they discovered the following results.

Theorem 1.3. *([BGP92, §7.8.1]) Let (M^n, d) be an Alexandrov space with curv $\geq k$. Then*

(1.3.1) The dimension of (M^n, d) is an integer or infinity, which is equal to the Hausdorff dimension;

(1.3.2) Let $T_p^-(M^n)$ be the cone over the space of directions of M^n at p, and let $sM^n = (M^n, s\,d)$ be the scaling of the space (M^n, d). Then $T_p^-(M^n)$ is isometric to the Gromov-Hausdorff limit of the pointed spaces $\{(sM^n, p)\}$ as $s \to +\infty$;

(1.3.3) The tangent cone $T_p^-(M^n)$ has curv ≥ 0;

(1.3.4) Let $\Sigma_p(M^n) = \{\vec{v} \in T_p^-(M^n) : |\vec{v}| = 1\}$. Then the unit tangent cone $\Sigma_p(M^n)$ has curv ≥ 1.

There are many non-smooth spaces with *curv* ≥ 0.

Example 1.4 (i) Let M^n be a complete smooth Riemannian manifold with *curv* $\geq k$. Suppose $\Omega \subset M^n$ is a convex sub-domain of M^n, where Ω is not necessarily smooth. Then, by a theorem of Buyalo, the boundary $\partial\Omega$ of Ω has *curv* $\geq k$ as well.

(ii) $M^2 = \{(x, y, z) \in \mathbb{R}^3 | z = \sqrt{x^2 + y^2}\}$ has *curv* ≥ 0, but $(0, 0, 0) = p$ is not a smooth point of M^2.

(iii) Let Ω be an American football. Its boundary $M^2 = \partial\Omega$ has *curv* ≥ 0.

(iv) If Σ is an Alexandrov space with *curv* ≥ 1 (e.g. $\Sigma = \mathbb{C}P^m$ is a complex projective space with the Fubini-Study metric), then we consider the cone over Σ as follows. Let $M = Cone_0(\Sigma) = \Sigma \times [0, +\infty)/\Sigma \times \{0\}$. For any pair of points

$\{(p,t_1),(q,t_2)\} \subset \Sigma \times [0,+\infty)$, we define

$$[d_M((p,t_1),(q,t_2))]^2 = t_1^2 + t_2^2 - 2t_1t_2\cos[d_\Sigma(p,q)]. \tag{1.6}$$

It was shown in [BGP92] that (M, d_M) above has $curv \geq 0$.

Recall that if for any length-minimizing geodesic $\sigma : [0,\ell] \to M^n$ of unit speed we have

$$\frac{d^2 f(\sigma(t))}{dt^2}(0) \leq c$$

in barrier sense then we say

$$Hess(f)(\sigma'(0),\sigma'(0)) \leq c$$

in barrier sense. With some additional efforts, one can easily verify the following estimate for the upper bound of Hessian of distance functions.

Proposition 1.5. *Let (M^n, d) be a complete Alexandrov space with curv $\geq k$. Suppose that $\varphi : [0,d] \to M^n$ is a length-minimizing geodesic of unit speed from x to p, and that $\sigma : [0,\epsilon] \to M^n$ is a quasi-geodesic of unit speed with $\angle_x(\sigma'(0),\varphi'(0)) = \frac{\pi}{2}$. Then, in barrier sense, for $d_p(x) = d(p,x)$ we have*

$$\mathrm{Hess}(d_p)(\sigma'(0),\sigma'(0)) \leq \begin{cases} \frac{1}{d_p}, & \text{if } k=0, \\ \cot(d_p), & \text{if } k=1, \\ \coth(d_p). & \text{if } k=-1. \end{cases} \tag{1.7}$$

Proof. It is clear that

$$\frac{d^2[h(u(t))]}{dt^2} = h'(u(t))u''(t) + h''(u(t))[u'(t)]^2. \tag{*}$$

Let $u(t) = d(p,\sigma(t))$. Thus, by our assumption we have $u'(0) = 0$.

(1) When $k = 0$, we let $h(u) = \frac{u^2}{2}$.

By definition, we see that if $f(t) = h(u(t))$ then $f''(t) \leq 1$. It follows that

$$1 \geq f''(t) = u(t)\mathrm{Hess}(d_p)(\sigma'(0),\sigma'(0)) + 0 = d_p(t)\mathrm{Hess}(d_p)(\sigma'(0),\sigma'(0)).$$

It follows that $\mathrm{Hess}(d_p)(\sigma'(0), \sigma'(0)) \le \frac{1}{d_p}$ for $k = 0$.

(2) When $k \ge 1$, we let $h(u) = 1 - \cos u$, $u(t) = d(p, \sigma(t))$ and $f(t) = h(u(t))$. It is known that $f''(t) \le [1 - f(t)]$. It follows that

$$-\frac{d^2[\cos d_p(t)]}{dt^2} \le \cos d_p(t). \tag{‡}$$

Therefore, by (*)-(‡) and initial condition $u'(0) = 0$, we have

$$[\sin d_p(t)]\mathrm{Hess}(d_p)(\sigma'(0), \sigma'(0)) \le \cos d_p(t).$$

(3) The case of $k = -1$ can be handled similarly. $\square$

When (M^n, d) has *curv* $\ge k > 0$, then the diameter is less than or equal to $\frac{\pi}{\sqrt{k}}$, i.e.

$$\mathrm{Diam}(M^n) \le \frac{\pi}{\sqrt{k}}. \tag{1.8}$$

Thus, the estimate (1.7) above makes sense for all $k > 0$ as well.

There is another way to see why the Hessian inequality (1.7) above holds in barrier sense, by the development maps used by Russian school of geometry, see §7.3 of Plaut's survey paper [Pl02, p861]. In fact, there are several equivalent definitions of quasi-geodesics.

Let us first recall a simple definition of quasi-geodesics without using development map.

Definition 1.6. *(Quasi-geodesics, [Pl02, p860]) Let Y be a metric space, $\eta : [a, b] \to Y$ be a Lipschitz curve of unit speed and let $d_q(t) = d(q, \eta(t))$. Then η is called a quasi-geodesic if for every $q \in Y$ there is a function h with $\lim_{s \to 0^+} \frac{h(s)}{s} = 0$, and $[d_q(t)^2 - t^2]'' \le h(d_q(t))$ for all $t \in [a, b]$. We write $[d_q(t)^2 - t^2]'' \le o(d_q(t))$ for short.*

Using the triangle comparison theorems for Alexandrov space with *curv* $\ge k$, one can show that

Proposition 1.7. *Let $q \in Y$, $\eta : [a, b] \to Y$ be a geodesic of unit speed and let $d_q(t) = d(q, \eta(t))$. Suppose that Y has curv $\geq k$ and that*

$$f(t) = \begin{cases} \frac{1}{2} d_q^2(t), & \text{if } k = 0, \\ \frac{1-\cos(\sqrt{k} d_q(t))}{k}, & \text{if } k > 0, \\ \frac{1-\cosh(\sqrt{-k} d_q(t))}{k}. & \text{if } k < 0. \end{cases}$$

Then

$$f''(t) \leq [1 - kf(t)]. \tag{1.9}$$

Moreover, the above condition holds if and only if the conclusion of Toponogov comparison theorem holds for any geodesic hinges in Y.

Perelman and Petrunin (cf. [PP94] and [PP96]) used the inequality (1.9) to define quasi-geodesics, see Definition 2.1 below as well.

The third definition of quasi-geodesics uses the development maps for a curve in the model space M_k^2, where M_k^2 is a complete simply-connected surface of constant curvature k. We thank Professor Stephanie Alexander for supplying us a correct definition of development maps.

Definition 1.8. *([AB96] Development map of a curve relative to p) Let σ_{xy} be a length-minimizing curve of unit speed in a metric space. Suppose γ is a rectifiable curve parameterized by arc-length in M and p is a point not on γ. The Alexandrov development $\tilde{\gamma}$ of γ from p is a curve in M_k^2 obtained as follows: associate to p a point $\tilde{p}$ in M_k^2, and associate to the minimizer $\sigma_{p\gamma(t)}$ to a minimizer $\tilde{\sigma}_{\tilde{p}\tilde{\gamma}(t)}$ of the same length, turning monotonically in t in such a way that t is also the arc-length parameter of $\tilde{\gamma}$. The union of the $\tilde{\sigma}_{\tilde{p}\tilde{\gamma}(t)}$ is called the "cone of the development". A development is possible whenever the maximum distance from p to γ is less than $\pi/\sqrt{\max\{0, k\}}$.*

We now state another equivalent definition of quasi-geodesics.

Proposition 1.9. *(page 861 of [Pl02] characterization of quasi-geodesics via development maps). Let Y be a complete Alexandrov space with curv $\geq k$ and let*

$c : [a,b] \to Y$ be a curve of unit speed. Suppose that $\tilde{c}^*$ is a k-development of c at p with $p \notin c([a,b])$. Then c is a quasi-geodesic if and only if c^* is k-convex in following sense:

(1.9.1) (Alexandrov) For any $p \in M$ and geodesic $\gamma : [a,b] \to M$, the k-development of γ at p is convex, where $p \notin \sigma([a,b])$ and M has curvature $\geq k$.

(1.9.2) Suppose that M has curvature $\geq k$. A curve c in M is a quasi-geodesic if and only if it has unit speed and its development of $\tilde{c}$ is k-convex relative to any point $p \notin c$.

It should be pointed out, for some non-smooth Alexandrov space (M^n, d), the distance function $d_p(x) = d(p,x)$ may *not* be convex for $x \in B_\varepsilon(p)$, where $B_\varepsilon(p) = \{q | d(p,q) < \varepsilon\}$. More precisely, the conclusion of the following Proposition for smooth Riemannian manifolds might fail for Alexandrov spaces.

Proposition 1.10. *Let (M^n, d) be a complete C^2-smooth Riemannian manifold with curv $\geq k$. Suppose that $\overline{\Omega}$ is a compact subset of M^n. Then there exists $\varepsilon_0 > 0$ depending only on $\overline{\Omega}$ such that $B_\varepsilon(p)$ is a convex subset for all $p \in \overline{\Omega}$ and $0 < \varepsilon \leq \varepsilon_0$.*

Here is a simple example of non-smooth Alexandrov space M^2 for which the conclusion of Proposition 1.10 fails.

Example 1.11. *Let $M^2 = \{(x,y,z) \in \mathbb{R}^3 | z = \sqrt{x^2+y^2}\}$ and $\overline{\Omega} = \{(x,y,z) \in M^2 | 0 \leq z \leq 1\}$. We choose a sequence of points $p_\varepsilon = (\varepsilon, 0, \varepsilon)$. It is clear that the metric disk $B_{2\varepsilon}(p_\varepsilon)$ is not convex for $0 < \varepsilon \leq 1$.*

The above example indicates that $d_{p_0} : B_\varepsilon(p_0) \to \mathbb{R}$ is not necessarily a convex function if there is a sequence of singular points $\{q_i\} \to p_0$ as $i \to +\infty$. In addition, the injectivity radius of M^n restricted to $\overline{\Omega}$ is zero since

$$\mathrm{inj}_{M^n}(p_\varepsilon) \leq 2\varepsilon \to 0$$

as $\varepsilon \to 0$.

In order to show that the distance function $d_p(x) = d(p, x)$ is free of critical points on $[B_\varepsilon(p) - \{p\}]$ in the sense of Gromov, Perelman cleverly constructed a local convex function $\psi_p : B_\delta(p) \to \mathbb{R}$ which is bi-Lipschitz comparable to d_p as follows.

Theorem 1.12. *(Perelman [Per94b], Kapovitch [Ka02, Lemma 4.2]) Let M^n be a finite dimensional Alexandrov space with curv $\geq k$. For each $p \in M^n$, there is a $\delta = \delta(p)$ depending only on the local volume growth of $B_\varepsilon(p)$ and there is a strictly convex non-negative function $\psi_p : B_\delta(p) \to [0, +\infty)$ with $\psi_p(p) = 0$ and*

$$B_{\frac{\varepsilon}{\lambda}}(p) \subset \psi_p^{-1}([0, \varepsilon]) \subset B_{\lambda\varepsilon}(p)$$

for some $\lambda \geq 1$ and $\varepsilon \in (0, \frac{\delta}{\lambda}]$. Consequently, the distance function d_p has no critical point on $[B_{\frac{\delta}{\lambda}}(p) - \{p\}]$ in the sense of Gromov.

Using Theorem 1.12 above, Perelman [Per94] further studied the local structure of Alexandrov spaces. In Example 1.4 above, we see that the cone over an Alexandrov space Σ^{n-1} with *curv* ≥ 1 has *curv* ≥ 0. In [BGP92], Burago-Gromov-Perelman also constructed parabolic or hyperbolic cones over a lower dimensional Alexandrov space Σ^m, the resulting cones have *curv* $\geq k_1$ for some other $k_1 \in \mathbb{R}$.

Definition 1.13. *(Perelman's MCS-spaces) We define MCS-spaces (spaces with multiple conic singularities) inductively on the dimensions.*

(0) A point $\{p\}$ is a 0-dimensional MCS-space;

(1) We say that M^n is an MCS-space if for each $p \in M^n$, there is a small ball $B_\varepsilon(p)$ of radius ε centered at p such that $B_\varepsilon(p)$ is homeomorphic to a cone over a lower dimensional MCS-space Σ.

A remarkable result of Perelman asserts that all possible singularities of any finite dimensional Alexandrov space M^n are at most as bad as conic singularities.

Theorem 1.14. *(Perelman [Per94b]) Let M^n be a complete finite dimensional Alexandrov space with curv $\geq k$. Then M^n must be an MCS-space.*

The proof of Perelman's structure theorem above is inspired by the celebrated Perelman's stability theorem.

Theorem 1.15. *(Perelman [Per94b], [Ka07], [BBI01,p400]) For any $k \in \mathbb{R}$ and any compact Alexandrov space M^n with curv $\geq k$, there is an $\varepsilon = \varepsilon(M^n) > 0$ such that every compact Alexandrov space Y^n with curv $\geq k$,*

$$d_{GH}(M^n, Y^n) < \varepsilon$$

and $\dim(M^n) = \dim(Y^n) = n$ must be homeomorphic to M^n, where $d_{GH}(X, Y)$ denotes the Gromov-Hausdorff distance between X and Y.

Consequently, $B_\varepsilon(p)$ is homeomorphic to a small ball $B_\varepsilon(O_p)$ in the tangent cone $T_p^-(M^n)$ for sufficiently small $\varepsilon > 0$.

For a non-compact complete space M^n, one consider $\{(M_i^n, p_i)\} \to (M^n, p)$ in the pointed Gromov-Hausdorff convergence. A Similar version of Theorem 1.15 for a sequence of pointed Alexandrov spaces also exists, see [Ka07].

Perelman's stability theorem can be used to simplify several crucial steps in Perelman's solution to Thurston's geometrization conjecture.

§2. Proof of the new sharp comparison theorem for quadrangles

In this section, we will provide a detailed proof of Theorem 0.2.

Let us first recall an equivalent definition of quasi-geodesics due to Perelman and Petrunin. Let

$$\rho_k(x) = \begin{cases} \frac{x^2}{2}, & \text{if } k = 0; \\ \frac{1}{k}[1 - \cos(\sqrt{k}x)], & \text{if } k > 0; \\ \frac{1}{k}[1 - \cosh(\sqrt{-k}x)], & \text{if } k < 0. \end{cases}$$

One considers

$$f_{p,\sigma,k}(t) = \rho_k(d(p, \sigma(t))). \tag{2.1}$$

Definition 2.1. *([PP94], [PP96]) Let (M^n, d) be a complete Alexandrov space with curv $\geq k$, and let $\sigma : [0, \ell] \to M^n$ be a Lipschitz curve of unit speed. If*

$$f''_{p,\sigma,k}(t) \leq 1 - k f_{p,\sigma,k}(t)$$

holds for any $p \in M^n$, then σ is called a quasi-geodesic segment.

We remark that one might be able to use the generalized 2nd fundamental form for curves to provide an equivalent definition of quasi-geodesics, see §4 below.

It is known that any geodesic segment is a quasi-geodesic. Petrunin also observed that comparison theorem holds for a class of quasi-geodesic hinges.

Proposition 2.2. *Let $\sigma_1 : [0, \ell_1] \to M^n$ be a length-minimizing geodesic and $\sigma_2 : [0, \ell_2] \to M^n$ be a quasi-geodesic with $p = \sigma_1(0) = \sigma_2(0)$ and*

$$\theta = \angle_p(\sigma_1'(0), \sigma_2'(0)).$$

(1) Suppose that M^n has curv ≥ 0. Then

$$[d(\sigma_1(\ell_1), \sigma_2(\ell_2))]^2 \leq \ell_1^2 + \ell_2^2 - 2\ell_1\ell_2 \cos\theta. \tag{2.2}$$

(2) Suppose that M^n has curv ≥ 1. Then

$$\cos[d(\sigma_1(\ell_1), \sigma_2(\ell_2))] \geq (\cos\ell_1)(\cos\ell_2) + (\sin\ell_1)(\sin\ell_2)\cos\theta.$$

Proof. (1) When $k = 0$, we let $f(t) = \frac{[d(\sigma_2(t), \sigma_1(\ell_1))]^2}{2}$. By an equivalent definition of quasi-geodesics, we have $f''(t) \leq 1$. Using the initial condition $f(0) = \frac{\ell_1^2}{2}$ and $f'(0) = \ell_1 \cos\theta$, we see that $f''(t) \leq 1$ implies

$$2f(t) \leq \ell_1^2 + t^2 - 2t\ell_1\cos\theta.$$

The inequality (2.2) follows.

(2) When $k = 1$, we will use an observation of Gromov to cancel the first derivatives. Let $h(t) = \cos[d(\sigma_2(t), \sigma_1(\ell_1))]$ and $h^*(t) = (\cos t)(\cos\ell_1) + (\sin t)(\sin\ell_1)\cos\theta$.

When $f(t) = 1 - h(t)$, by an equivalent definition of quasi-geodesics, we have $f''(t) \leq [1 - f(t)]$. It follows that

$$h''(t) \geq h(t).$$

Inspired by Gromov, we let

$$\eta(t) = h'(t)h^*(t) - [h^*(t)]'h(t).$$

By the inequality $h''(t) \geq h(t)$, we see that

$$\eta'(t) \geq 0,$$

whenever $\min\{h(t), h^*(t)\} \geq 0$.

By our assumption, we see that $\eta(0) = 0$. It follows from $\eta'(0) \geq 0$ that $\eta(t) \geq 0$. Whenever $\min\{h(t), h^*(t)\} \geq 0$, we have $\{\log[h(t)]\}' \geq \{\log[h^*(t)]\}'$ and hence

$$h(t) \geq h^*(t).$$

The other cases could be similarly, we leave it to readers. In fact, the differential inequality $f''(t) \leq [1 - f(t)]$ with the initial conditions above would implies $h(t) \geq h^*(t)$, see textbook [Pete98], page 327-330. $\square$

Proof of Theorem 0.2.

We first verify Theorem 0.2 (2).

Let $\alpha = \angle_q(\sigma'_{q\hat{p}}(0), \sigma'_{q\hat{q}}(0))$, $\tilde{q} = \sigma_{q\hat{q}}(L\cos\alpha)$ and $L = d(q, \hat{p})$. Since M^n has $curv \geq 0$ and $\sigma_{q\hat{p}}$ is length-minimizing, we have

$$[d(\hat{p}, \tilde{q})]^2 \leq L^2 + (L\cos\alpha)^2 - 2(L\cos\alpha)^2 = (L\sin\alpha)^2. \tag{2.3}$$

Let $T = d(p, q)$. Applying comparison theorem twice to the geodesic triangle $\triangle_1$, we also obtain

$$L^2 \leq [T^2 + s^2 - 2sT\cos\theta] \tag{2.4}$$

and

$$
\begin{aligned}
s^2 &\leq \left[L^2 + T^2 - 2LT\cos\left(\frac{\pi}{2} - \alpha\right)\right] \\
&= L^2 + T^2 - 2LT\sin\alpha. \qquad (2.5)
\end{aligned}
$$

Using (2.3)-(2.5), we have

$$
\begin{aligned}
d(\hat{p}, \tilde{q}) &\leq L\sin\alpha \\
&\leq \frac{1}{2T}[L^2 + T^2 - s^2] \\
&\leq \frac{1}{2T}[(T^2 + s^2 - 2sT\cos\theta) + (T^2 - s^2)] \\
&\leq \frac{1}{2T}[2T^2 - 2sT\cos\theta] \\
&= T - s\cos\theta.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d(\hat{p}, \sigma_{q\hat{q}}(\mathbb{R})) &\leq d(\hat{p}, \tilde{q}) \\
&\leq T - s\cos\theta.
\end{aligned}
$$

This completes the proof of Theorem 0.2 (2).

To see that the first assertion of Theorem 1.2 is true, we use the development maps of $\sigma_{p\hat{p}}$ and $\sigma_{q\hat{q}}$ relative to the midpoint $q_{mid} = \sigma_{pq}(\frac{T}{2})$.

We see that, by (1.9.2) that $\sigma^*_{p,\hat{p}}$ lies *inside* of the trapezoid in the model space $M^2_0 = \mathbb{R}^2$. We now use the fact $\sigma^*_{p,\hat{p}}$ has unit speed to conclude that

$$
d(\sigma^*_{p,\hat{p}}(t), \sigma^*_{q\hat{q}}(\mathbb{R})) \leq d(\varphi^*(t'), \sigma^*_{q\hat{q}}(\mathbb{R})) \qquad (2.6)
$$

with $\frac{t'}{t} \to 1$ as $t \to 1$, where $\varphi^* : [0, \infty) \to \mathbb{R}^2$ is a straight line with $\varphi^*(0) = p$ and $[\varphi^*]'(0) = [\sigma^*_{p,\hat{p}}]'(0)$.

In the model space, we have

$$
d(\varphi^*(t'), \sigma^*_{q\hat{q}}(\mathbb{R})) \leq T - t'\cos\theta. \qquad (2.7)
$$

It follows from (2.6)-(2.7) and discussion above that the inequality (0.3) holds in barrier sense.

The equality case in (0.4) holds if and only if the four points span a totally geodesic flat trapezoid in M^n. $\square$

In his important preprint [Per91], Perelman showed that, for fixed θ, one has

$$d(\hat{p}, \sigma_{q\hat{q}}(\mathbb{R})) \le T - s\cos\theta + o(s^2),$$

where $\lim_{s\to 0} \frac{o(s^2)}{s^2} = 0$.

We present some direct applications of Theorem 0.2 and its proof to the equidistance evolutions.

Definition 2.3. *Let Ω be a subset of a complete Alexandrov space M^n with curv $\ge k$.*

(1) If for any pair of points $\{p, q\} \subset \Omega$ there is a length-minimizing geodesic $\sigma_{p,q} : [0, \ell] \to M^n$ from p to q such that $\sigma((0, \ell)) \subset \Omega$, then Ω is called convex.

(2) Let Ω be a convex domain and let $\overline{\Omega}$ be its closure. If any quasi-geodesic $\sigma : [0, \ell] \to M$ tangent to the boundary $\partial\Omega$ of Ω at $\sigma(0)$ has the property that $\sigma(t) \notin \overline{\Omega}$ for $t \in (0, \delta]$ and some positive $\delta \le \ell$, then Ω is called a **strictly** *convex domain.*

(3) Let $\Omega \subset U$. If for any pair of points $\{p, q\} \subset \Omega$ and any length-minimizing geodesic $\sigma_{p,q} : [0, \ell] \to U$ from p to q such that $\sigma((0, \ell)) \subset \Omega$, then Ω is called totally convex relative to U.

Corollary 2.4. *Let M^n be a complete Alexandrov space with curv ≥ 0. Suppose that Ω_0 is a compact convex domain in M^n and $\partial\Omega_0$ is strictly convex. Then $\Omega_{-T} = \{p \in \Omega_0 | d(p, \partial\Omega_0) \ge T\}$ is strictly convex for all $T < T_0 = \max\{d(p, \partial\Omega_0) | p \in \Omega_0\}$. Furthermore, Ω_{-T} is totally convex in Ω_0, $\dim[\Omega_{-T_0}] = 0$ and Ω_0 is contractible.*

Remark: For the special case when M^n is a complete smooth Riemannian manifold with non-negative curvature but positive curvature outside a compact set, Corollary 2.4 was implicitly proved by Cheeger-Gromoll, see page 421 and page 431 of [CG72].

Proof of Corollary 2.4. We present a proof inspired by Cheeger-Gromoll [CG72] with some modifications and will use the proof of Theorem 0.2.

Let $f : M^n \to \mathbb{R}$ be a signed distance as follows: $f(x) = d(x, \partial\Omega)$ for $x \in \Omega$ and $f(x) = -d(x, \partial\Omega)$ for $x \notin \Omega$.

Step 1. Proof of weak convexity.

We first consider the signed distance function $f(x)$ restricted to a length-minimizing geodesic segment in the same way as Cheeger-Gromoll did in [CG72]. Let $\sigma : [0, \ell] \to \Omega$ be a length-minimizing geodesic of unit speed. For each interior point p of Ω, we let $\Lambda_{p,\partial\Omega} = \{\vec{v} \in T_p^-(M) \quad | \quad \vec{v} = \varphi'(0), \varphi : [0, f(p)] \to M, |\varphi'(t)| = 1, \varphi(0) = p, \varphi(f(p)) \in \partial\Omega\}$ be the subset of all unit minimizing directions from p to Ω. Since $Min_{p,\partial\Omega}$ is a compact subset of Σ_p, there are $q \in \partial\Omega$ and $\vec{w} = \sigma'_{pq}(0) \in \Lambda_{p,\partial\Omega}$ such that

$$\theta = \angle_p(\sigma'(0), \vec{w}) = \inf\{\angle_p(\sigma'(0), \vec{v}) \quad | \quad \vec{v} \in Min_{p,\partial\Omega}\}$$

and

$$d(p, q) = f(p) = d(p, \partial\Omega)$$

hold.

Let us verify the following.

Claim 2.5. Let $p \in int(\Omega)$, $q \in \partial\Omega$, $f : M \to \mathbb{R}$, θ and $\sigma : [0, \ell] \to \Omega$ be as above. Then, for each $s > 0$, there is a quasi-geodesic $\psi : [0, \delta] \to M$ tangent to $\partial\Omega$ at $\psi(0) = q$ such that

$$f(\sigma(s)) = d(\sigma(s), \partial\Omega) < d(\sigma(s), \psi(t_s)) \leq f(p) - s\cos\theta \tag{2.8}$$

holds for some $t_s > 0$.

Recall that $d(p, q) = d(p, \partial\Omega)$, by the first variational formula, our length-minimizing geodesic σ_{pq} is orthogonal to $\partial\Omega$ at q. As Perelman observed that the tangent cone $T_q^-(\overline{\Omega})$ has metric splitting

$$T_q^-(\overline{\Omega}) = [0, \infty) \times T_q^-(\partial\Omega). \tag{2.9}$$

We now choose a length-minimizing geodesic segment from q to $\sigma(s)$ of unit speed, say $\sigma_{q\sigma(s)} : [0, L] \to \Omega$. Let $\vec{\xi} = \sigma'_{q\sigma(s)}(0)$. Using (2.9), we can write

$$\vec{\xi} = (\cos\alpha)\vec{\eta} + (\sin\alpha)\vec{w}^* \tag{2.10}$$

for some $\vec{\eta} \in T_q^-(\partial\Omega)$ and $0 < \alpha < \frac{\pi}{2}$, where $\vec{w}^*$ is the left derivative of σ_{pq} : $[0, d(p,q)] \to M$ at its endpoint q and $|\vec{\eta}| = 1 = |\vec{w}^*|$.

For any unit direction $\vec{\eta}$, there is a quasi-geodesic $\psi : [0, \delta] \to M$ with $\psi'(0) = \vec{\eta}$. Let $t_s = L\cos\alpha$ where $L = d(q, \sigma(s))$. By the proof of Theorem 0.2, we have

$$d(\sigma(s), \psi(t_s)) \leq d(p,q) - s\cos\theta. \tag{2.11}$$

Because $t_s = L\cos\alpha > 0$ and our domain Ω is strictly convex, by definition 2.3 (2), we have

$$f(\sigma(s)) = d(\sigma(s), \partial\Omega) < d(\sigma(s), \psi(t_s)). \tag{2.12}$$

Claim 2.5 follows from (2.11)-(2.12).

It now follows from Claim 2.5 that $f(x) = d(x, \partial\Omega)$ is a concave function for $x \in \Omega$.

Step 2. Proof of strict convexity.

Let $\Omega_{-c} = f^{-1}([c, \infty))$ for $c \geq 0$. We would like to show that the convex domain Ω_{-c} is strictly convex for $c > 0$ by using (2.8) and a theorem of Perelman-Petrunin. Let $\phi : [0, \delta^*] \to M$ be a quasi-geodesic segment tangent to $\partial\Omega_{-c}$ at p. Our goal is to verify

$$f(\phi(s)) < f(\phi(0)) \tag{**}$$

for any $s \in (0, \delta]$ and some $\delta > 0$.

For special case when the above quasi-geodesic $\phi : [0, \delta^*] \to M$ is a length-minimizing geodesic segment, the inequality $f(\phi(s)) < f(\phi(0))$ is a direct consequence of (2.8) by choosing $\theta = \frac{\pi}{2}$.

For the general case of tangential quasi-geodesic $\phi : [0, \delta^*] \to M$, we use Corollary 3.3.3 of [Petr07] to get the fact that the function

$$\eta(s) = f(\phi(s)) \tag{2.13}$$

is a concave function of s with $\eta'(0) = 0$. It follows from concavity and $\eta'(0) = 0$ that

$$\eta(s) = f(\phi(s)) \leq f(\phi(0)) \tag{2.14}$$

for all $s \geq 0$.

If $f(\phi(s_0)) < f(\phi(0))$ for some $s_0 > 0$, by concavity we have that

$$f(\phi(s)) \leq f(\phi(s_0)) + (s - s_0)\frac{f(\phi(s_0)) - f(\phi(0))}{s_0} \leq f(s_0), \tag{2.15}$$

for all $s \geq s_0$.

Hence, we may assume that there were $s_0 > 0$ such that

$$f(\phi(s_0)) = f(\phi(0)) \tag{2.16}$$

and we will derive a contradiction as follows.

Choose a length-minimizing geodesic $\sigma_{pq} : [0, c] \to M$ from p to $\partial\Omega$ such that $q \in \partial\Omega$, $d(p, q) = c = d(p, \partial\Omega)$ and

$$\angle_p(\phi'(0), \sigma'_{pq}(0)) = \inf\{\angle_p(\phi'(0), \vec{w}) \quad | \quad \vec{w} \in Min(p, \partial\Omega)\} = \frac{\pi}{2}.$$

We further choose the midpoint $q_{mid} = \sigma_{pq}(\frac{c}{2})$. We will consider the development map of the quasi-geodesic ϕ relative to q_{mid}. Since q_{mid} is an interior point of a length-minimizing geodesic σ_{pq}, its log image $log_p(q_{mid})$ is unique, which is equal to $c\sigma'_{pq}(0) = \tilde{q}_{mid}$. Let us consider a "half-space" in the tangent cone $T_p(M)$ relative to $\tilde{q}_{mid}$ as follows:

$$Half_{p,\tilde{q}_{mid}} = \{\vec{u} \in T_p^-(M) \quad | \quad \langle \vec{u}, \tilde{q}_{mid} \rangle \geq 0\}.$$

Because $\phi : [0, s_0] \to M$ is a quasi-geodesic, its "cone of development" relative q_{mid} can be isometrically embedded into $Half_{p,\tilde{q}_{mid}}$ with vertex $\tilde{q}_{mid}$. It follows that there is $\vec{u}_0 \in log_p(\phi(s_0))$ such that

$$\theta_0 = \angle_p(\vec{u}_0, \tilde{q}_{mid}) \leq \frac{\pi}{2}. \tag{2.17}$$

Let $\hat{p} = \phi(s_0)$, $\hat{s}_0 = d(p, \phi(s_0)) = |\vec{u}_0|$ and let $\sigma_{p\hat{p}} : [0, \hat{s}_0] \to M$ be a length-minimizing geodesic from p to p_0 such that

$$\sigma'_{p\hat{p}}(0) = \frac{\vec{u}_0}{|\vec{u}_0|}.$$

We now apply inequality (2.8) to conclude that

$$f(\phi(s_0)) = f(\sigma_{p\hat{p}}(\hat{s}_0)) < f(p) - \hat{s}_0 \cos\theta_0 \leq f(p).$$

This completes the proof of the fact that Ω_{-c} is strictly convex. $\square$

We emphasize that the equidistance evolution plays an important role in Corollary 2.4. The conclusion of Corollary 2.4 fails if we replace the distance function $f(x) = d(x, \partial\Omega_0)$ by an arbitrary concave function $h : \Omega_0 \to [0, T_0]$.

For example, let $\Omega_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $\Omega_{T_0} = \{(x, 0) : |x| \leq \frac{1}{2}\}$. We can construct a convex function $\psi : \Omega_0 \to [-T_0, 0]$ such that $\psi^{-1}(0) = \Omega_0$ and $\psi^{-1}(-T_0) = \Omega_{-T_0}$. Choose $h(x) = -\psi(x)$. We notice that $h^{-1}([T, T_0]) = \Omega_{-T}$ is convex but not necessarily strictly convex. For instance, $\Omega_{-T_0} = h^{-1}(T_0)$ is *not* strictly convex and $\dim(\Omega_{-T_0}) = 1 > 0$. Thus, the conclusion of Corollary 2.4 fails for non-distance function h.

We also remark that if the assumption of non-negative curvature is removed, then the conclusion of Corollary 2.4 fails. Here is an example.

Example 2.6. Let $M^2 = \{(x, y, z) | x^2 + y^2 - z^2 = 1\}$ be a one-sheet hyperboloid in $\mathbb{R}^3$. It is clear that M^2 has negative curvature. Let $\Omega_0 = \{(x, y, z) \in M^2 | z \leq 0\}$. It is clear that $\partial\Omega_0$ is a closed geodesic. Hence Ω_0 is a convex subset of M^2. However, $\Omega_{-T} = \{p \in \Omega_0 | d(p, \partial\Omega_0) \geq T\}$ has strictly concave boundary $\partial\Omega_{-T}$ for $T > 0$. Thus Ω_{-T} is no longer a convex subset of M^2 for $T > 0$.

§3 Further applications of sharp quadrangle comparisons

In this section, we present two more applications of our new sharp quadrangle comparisons.

First, we present a direct proof of Perelman's soul construction theorem. His original proof used a contradiction argument.

Theorem 3.1. *(Perelman [Per91,§6]) Let M^n be a complete Alexandrov space with curv $\geq k \geq 0$. Suppose that Ω_0 is a convex sub-domain of M^n. Then $f(x) = d(x, \partial\Omega_0)$ is a concave function for $x \in \Omega_0$. Moreover, if $k > 0$, then $\Omega_c = f^{-1}([c, +\infty))$ is strictly convex for $c > 0$.*

Proof. We first consider the case of $k = 0$, so *curv* ≥ 0. For each $p \in \Omega_0$ with $d(p, \partial\Omega_0) > 0$ and for each length-minimizing geodesic

$$\sigma_{p\hat{p}} : [0, s] \to \Omega_0$$

of unit speed, we would like to show that

$$t \mapsto f(\sigma_{p\hat{p}}(t)) = \varphi_\sigma(t)$$

is a concave function at $t = 0$.

The derivative of φ_σ is related to the angle θ between $\sigma'_{p\hat{p}}(0)$ and $\mathrm{Min}(p, \partial\Omega_0)$, where

$$\begin{aligned} \mathrm{Min}(p, \partial\Omega_0) = \{\sigma'_{pq}(0) | \sigma_{pq} : [0, T] \to \Omega_0 & \text{ is length-minimizing and of unit speed} \\ & \text{from } p \text{ to } q \in \partial\Omega_0 \text{ with } d(p, q) = d(p, \partial\Omega_0)\}. \end{aligned}$$

Let

$$\theta = \min\{\angle_p(\sigma'_{p\hat{p}}(0), \vec{w}) | \vec{w} \in \mathrm{Min}(p, \partial\Omega_0)\}.$$

We can show that

$$\frac{d^+\varphi_\sigma}{dt}(0) = -\cos\theta. \tag{3.1}$$

By the proof of Corollary 2.4, we see that

$$d(p, \partial\Omega_0) \leq d(p, q) - s\cos\theta, \tag{3.2}$$

where $s = d(p, \hat{p})$. It follows that

$$f(\sigma_{p\hat{p}}(s)) \leq f(\sigma_{p\hat{p}}(0)) + \frac{d^+[f \circ \sigma_{p\hat{p}}]}{ds}(0)\, s. \tag{3.3}$$

Therefore, $f(x) = d(x, \partial\Omega_0)$ is a concave function for the case of $curv \geq 0$. When $curv \geq k > 0$, the proof of Corollary 2.4 implies that

$$d(\hat{p}, \partial\Omega_0) < d(p, q) - s\cos\theta \tag{3.4}$$

if $s > 0$ and $0 < \theta < \pi$. Equivalently, we have

$$f(\sigma_{p\hat{p}}(s)) < f(\sigma_{p\hat{p}}(0)) + \frac{d^+[f \circ \sigma_{p\hat{p}}]}{ds}(0)\, s, \tag{3.5}$$

when $s > 0$ and $0 < \theta < \pi$. Notice that if $\sigma'_{p\hat{p}}(0)$ is tangent to $\partial\Omega_c = f^{-1}(c)$ with $c = f(p)$, then $\theta = \frac{\pi}{2}$. It follows from (3.5) that Ω_c is strictly convex for any $c > 0$. $\square$

Our second application is to improve Petrunin's second variational formula for length in Alexandrov spaces of $curv \geq 0$.

Let us begin with convex curves in a 2-dimensional Alexandrov space with $curv \geq 0$.

Proposition 3.2. *Let M^2 be a complete Alexandrov surface with $curv \geq 0$. Suppose that Ω_0 is a convex sub-domain of M^2 and $\Omega_c = \{x \in \Omega_0 | d(x, \partial\Omega_0) \geq c\}$. Then the length function $t \mapsto L(\partial\Omega_t)$ is a non-increasing function for $t \geq 0$.*

Proof. We have shown that $f(x) = d(x, \partial\Omega_0)$ is a concave function. There is a Sharafutdinov semi-flow for gradient of f, see [KPT07]. It was observed by Petrunin that the gradient flow

$$\frac{d^+\sigma}{dt} = \frac{\nabla f}{|\nabla f|^2}\Big|_{\sigma(t)} \tag{3.6}$$

is a distance non-increasing map. Such a gradient flow induces a Sharafutdinov projection $\partial\Omega_c \to \partial\Omega_{c+t}$ for $c \geq 0$ and $t \geq 0$. It is known that Sharafutdinov projection is distance non-increasing. Thus we conclude that

$$L(\partial\Omega_{c+t}) \leq L(\partial\Omega_c) \tag{3.7}$$

for $c \geq 0$ and $t \geq 0$. $\square$

We now would like to elaborate the idea above. In fact, we can refine the analysis above point-wise as follows. Let

$$\pi_{c,c+t} : \partial\Omega_c \to \Omega_{c+t}$$

be the Sharafutdinov distance non-increasing projection, where $\dim[\Omega_c] = \dim(M^2) = 2$. For each curve $\sigma_{c+t} \subset \partial\Omega_{c+t}$, we let

$$\sigma_c = \pi^{-1}_{c,c+t}(\sigma_{c+t}).$$

Corollary 3.4. *Let M^2, Ω_0, Ω_c, $\pi_{c,c+t}$, σ_{c+t} and σ_c be as above. Then*

$$L(\sigma_{c+t}) \leq L(\sigma_c) \tag{3.8}$$

for any $c \geq 0$ and $t \geq 0$.

We now consider the case when the initial curve σ_0 is a geodesic segment. Let us give a short proof of Petrunin's 2nd variational formula for this special case.

Theorem 3.5. *(Improved Petrunin's formula for 2-dimensional case) Let M^2 be a complete Alexandrov surface with non-negative curvature and Ω_0 be a compact convex domain. Suppose that $\hat{\sigma} : [-\varepsilon, L_0 + \varepsilon] \to M^2$ is a length-minimizing geodesic of unit speed with $\hat{\sigma}([-\varepsilon, L_0 + \varepsilon]) \subset \partial\Omega_0$, $f(x) = d(x, \partial\Omega_0)$ and $\varphi_q : [0, \delta] \to M^n$ is the gradient semi-flow pointing inside Ω_0 with*

$$\begin{cases} \frac{d^+\varphi_q}{dt} = \left.\frac{\nabla f}{|\nabla f|^2}\right|_{\varphi_q(t)}, \\ \varphi_q(0) = q \in \partial\Omega_0, \end{cases}$$

and suppose that $\sigma_t : [0, L_0] \to M^n$ *is given by* $\sigma_t(s) = \varphi_{\hat{\sigma}(s)}(t)$. *Then*

$$\frac{d^2[L(\sigma_t)]}{dt^2}(0) \le 0, \tag{3.10}$$

where $L(\sigma_t)$ *denotes the length of* σ_t.

Proof. Since $\sigma_0(s) = \hat{\sigma}(s)$ and $\hat{\sigma} : [0, \ell] \to M^n$ is a length-minimizing geodesic, we have

$$\frac{d[L(\sigma_t)]}{dt}(0) = 0. \tag{3.11}$$

In addition, we have shown above that $t \mapsto L(\sigma_t)$ is a non-increasing function of t. Thus

$$\frac{d[L(\sigma_t)]}{dt}(t) \le 0. \tag{3.12}$$

Therefore, we have

$$\frac{d^2[L(\sigma_t)]}{dt^2}(0) = \lim_{t\to 0} \frac{\frac{d^+[L(\sigma_t)]}{dt} - \frac{d^+[L(\sigma_0)]}{dt}}{t} \le 0.$$

This completes the proof. $\square$

The 2nd variational formula for lengths in higher dimensional Alexandrov spaces will be discussed elsewhere. For the earlier work in this direction, see Petrunin's paper [Petr98].

§4 Two open problems in Alexandrov's geometry and possible approaches

In this section, we discuss two open problems in Alexandrov's geometry along with possible approaches. The first one is about the curvature bound of the boundary of a convex domain in an Alexandrov space. The second one is related to the geodesic semi-flow on Alexandrov spaces.

The following is a well-known problem in Alexandrov's geometry.

Open Problem 4.1. *Let M^n be a complete Alexandrov space with curv $\geq k$ and Ω be a convex domain of M^n with* $\dim(\Omega) = \dim(M^n)$. *Prove that the boundary $\partial\Omega$ of Ω has curv $\geq k$ with respect to the intrinsic metric of $\partial\Omega$.*

Earlier work for non-smooth convex domains in a smooth Riemannian manifold M^n with *curv* $\geq k$ was carried out by Buyalo [Bu79]. Recently, Alexander-Kapovitch-Petrunin ([AKP07]) showed that $(\partial\Omega, d_{\partial\Omega})$ has *curv* $\geq k$ globally, i.e. the conclusion of Toponogov comparison theorem holds for "large" geodesic triangles in $\partial\Omega$ as well.

In Example 1.2 of an important paper [PP94], Perelman and Petrunin implicitly stated the following interesting result: *"Let Ω_0 be a convex domain in a complete Alexandrov space M^n with curv $\geq k$, and $\overline{\Omega}_0$ be its closure in M^n. Suppose that $\sigma : [0, \ell] \to \partial\Omega_0$ is a length-minimizing geodesic segment of unit speed with respect to the intrinsic metric $d_{\partial\Omega_0}$. The curve σ is necessarily a quasi-geodesic segment in $\overline{\Omega}_0$ (or in its doubling $[\overline{\Omega}_0 \cup_{\partial\Omega_0} \overline{\Omega}_0]$).*

One might take a different approach to the above Open Problem as follows.

Modified Problem 4.1.A. *Prove that Toponogov comparison theorem holds for intrinsic geodesic triangles in $\partial\Omega_0$.*

In particular, if M^n has curv ≥ 0 and if $\Delta \subset \partial\Omega_0$ is a triangle whose sides are length-minimizing segments with respect to the intrinsic metric of $\partial\Omega_0$, then is it true that the total (intrinsic) inner angles of Δ greater than or equal to π?

In order to carry out our proposed approach above, we might want to use an alternative definition of quasi-geodesics by introducing generalized 2nd fundamental form for any subsets in an Alexandrov space M^n.

Following Perelman-Petrunin's notion [Petr98], we say $\vec{v} = \log_p q$ if there is a shortest path from p to q in M^n which tangent to $\vec{v}$ and has length $|\vec{v}| = d_{M^n}(p, q)$. It is known that $\log_p : M^n \to T_p^-(M^n)$ is a distance non-decreasing map by "global" Toponogov comparison theorem.

For $q \neq p$ and $\vec{v} \in T_p^-(M^n)$ with $|\vec{v}| = 1$, we let

$$\angle_p(\vec{v}, q) = \min\{\angle_p(\vec{v}, \vec{w}) | \vec{w} \in \log_p(q)\}.$$

For a C^2-smooth submanifolds A in a smooth Riemannian manifold M^n, the second fundamental form can be reviewed as follows. If $\vec{v} \perp T_p(A)$ with $|\vec{v}| = 1$ and if $\sigma : (-\varepsilon, \varepsilon) \to A$ is a smooth curve of unit speed with $\sigma(0) = p$, then

$$\begin{aligned} \mathrm{II}_A^{\vec{v}}(\sigma'(0), \sigma'(0)) &= -\langle \nabla_{\sigma'}\sigma', \vec{v} \rangle|_{t=0} \\ &= \lim_{\varepsilon \to 0} -\frac{1}{\varepsilon}[\langle \vec{v}, \exp_p^{-1}\sigma(\varepsilon)\rangle + \langle \vec{v}, \exp_p^{-1}\sigma(-\varepsilon)\rangle]. \end{aligned} \tag{4.1}$$

For example, let $M^2 = \mathbb{R}^2$ and $A = S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. We consider $\sigma(t) = (\cos t, \sin t)$, $\vec{v} = (1, 0)$. By the above formula, we have

$$\mathrm{II}_A^{\vec{v}}(\sigma'(0), \sigma'(0)) = 1 > 0.$$

Inspired by discussion above and (4.1), we consider the generalized 2nd fundamental form for $A \subset M^n$ in the barrier sense.

Definition 4.2. *Let M^n be a complete Alexandrov space, $p \in A \subset M^n$ and $\vec{v} \in T_p^-(M^n)$ with $|\vec{v}| = 1$. We let*

$$\theta_{p,\vec{v}}(\varepsilon) = \inf\{\angle_p(\vec{v}, q) | q \in [A - B_\varepsilon(p)]\}.$$

If

$$\lim_{\varepsilon \to 0^+} -\frac{\cos \theta_{p,\vec{v}}}{\varepsilon} \leq 0 \tag{4.2}$$

holds, then we say that the subset A is concave relative to $\vec{v}$ at p.

Using our new definition above, we are led to study the following problem.

Sub-Problem 4.1.B. *(1) Let $\sigma : [0, \ell] \to M^n$ be a quasi-geodesic segment of unit speed in M^n and $A = \sigma([0, \ell])$. Prove that the quasi-geodesic $A = \sigma([0, \ell])$ is concave relative to any unit direction $\vec{v} \in T_{\sigma(t)}^-(M^n)$.*

(2) Let $\sigma : [0, \ell] \to M^n$ be a length-minimizing geodesic segment of unit speed, $p = \sigma(t_0)$ with $0 < t_0 < \ell$ and $\vec{v} \perp \sigma'(t_0)$. Prove that

$$\lim_{\varepsilon \to 0^+} \frac{\theta_{p,\vec{v}}(\varepsilon)}{\varepsilon} = 0. \tag{4.2}$$

Our next question is related to the quasi-geodesic semi-flow on Alexandrov spaces. In an earlier work of Perelman-Petrunin, quasi-geodesic segments were extended in a "non-unique" way. This was due the fact that the choice of polar vectors are not unique.

Definition 4.3. *Let M^n be a complete Alexandrov space with curv $\geq k$. Two unit tangent vectors $\{\vec{v}_1, \vec{v}_2\} \subset T_p^-(M^n)$ are said to be polar if*

$$\cos \angle_p(\vec{v}_1, \vec{w}) + \cos \angle_p(\vec{v}_2, \vec{w}) \geq 0 \tag{4.4}$$

for any $\vec{w} \in T_p^-(M^n)$ with $|\vec{w}| = 1$.

Recall that

$$\cos \theta_1 + \cos \theta_2 = 2 \cos \frac{\theta_1 + \theta_2}{2} \cos \frac{\theta_1 - \theta_2}{2}. \tag{4.5}$$

Because the unit tangent cone $\Sigma_p(M^n)$ has *curv* ≥ 1, its diameter is $\leq \pi$. The polar condition (4.4) is equivalent to

$$\angle_p(\vec{v}_1, \vec{w}) + \angle_p(\vec{v}_2, \vec{w}) \leq \pi \tag{4.6}$$

for all $\vec{w} \in \Sigma_p^-(M^n)$,

When the diameter of $\Sigma_p(M^n)$ is $\leq \frac{\pi}{2}$, for any given $\vec{v}_1 \in \Sigma_p(M^n)$, there are infinitely many $\vec{v}_2 \in \Sigma_p(M^n)$ polar to $\vec{v}_1$.

However, if the radius of $\vec{v}_1$ in $\Sigma_p(M^n)$ satisfies

$$\text{rad}(\vec{v}_1) = \max\{\angle_p(\vec{v}_1, \vec{w}) | \vec{w} \in \Sigma_p(M^n)\} > \frac{\pi}{2}, \tag{4.7}$$

then there is a unique canonical choice of $\vec{v}_2$ which is polar to $\vec{v}_1$ with $\angle_p(\vec{v}_1, \vec{v}_2) = \text{rad}(\vec{v}_1)$.

Proposition 4.5. *Let M^n be a complete Alexandrov space with curv $\geq k$, $\vec{v}_1 \in \Sigma_p(M^n)$ has* $\mathrm{rad}(\vec{v}_1) > \frac{\pi}{2}$. *Then*

$$\Omega_{\frac{\pi}{2}+\varepsilon} = \{\vec{w} \in T_p(M^n) | \angle_p(\vec{v}_1, \vec{w}) \geq \frac{\pi}{2} + \varepsilon\}$$

has strictly convex boundary $\partial\Omega_{\frac{\pi}{2}+\varepsilon}$ in $\Sigma_p(M^n)$ for any $0 < \varepsilon < [\mathrm{rad}(\vec{v}_1) - \frac{\pi}{2}]$. Consequently, there is a unique unit vector $\vec{v}_2 \in \Sigma_p(M^n)$ with $\angle_p(\vec{v}_1, \vec{v}_2) = \mathrm{rad}(\vec{v}_1)$.

Proof. In fact, $\vec{v}_2$ is the unique soul point of $\Omega_{\frac{\pi}{2}+\varepsilon}$, where we used the fact that

$$\mathrm{Hess}(f) \leq \cot(f) + df \otimes df \tag{4.8}$$

in the barrier sense, where $f(\vec{w}) = d_\Sigma(\vec{v}_1, \vec{w})$. Thus $\Omega_{\frac{\pi}{2}} = f^{-1}([\frac{\pi}{2}, \pi])$ is convex and $\Omega_{\frac{\pi}{2}+\varepsilon} = f^{-1}([\frac{\pi}{2} + \varepsilon, \pi])$ is strongly convex for $\varepsilon > 0$. $\square$

In order to see how polar vectors are related to quasi-geodesics, we recall a result of Perelman-Petrunin.

Proposition 4.6. *([PPe 94]) Let $\sigma_1 : [0, \ell_1] \to M^n$ and $\sigma_2 : [\ell_1, \ell_2] \to M^n$ be two quasi-geodesic segments of unit speed with $\sigma_1(\ell_1) = \sigma_2(\ell_1)$. Suppose that $\vec{v}_1 = \frac{d^-\sigma_1}{dt}(\ell_1)$ is polar to $\vec{v}_2 = \frac{d^+\sigma_2}{dt}(\ell_1)$, where*

$$\frac{d^\pm\sigma}{dt}(t) = \lim_{\varepsilon\to 0^+} \frac{\log_{\sigma(t)} \sigma(t \pm \varepsilon)}{\varepsilon}. \tag{4.9}$$

Then $\sigma_1 \cup \sigma_2$ forms an extended quasi-geodesic segment, where

$$(\sigma_1 \cup \sigma_2)(t) = \begin{cases} \sigma_1(t), & \text{if } 0 \leq t \leq \ell_1; \\ \sigma_2(t), & \text{if } \ell_1 \leq t \leq \ell_2. \end{cases}$$

Inspired by Propositions 4.4-4.6, we introduce the notion of canonical quasi-geodesics.

Definition 4.7. *Let $\sigma : [0, \ell] \to M^n$ be a quasi-geodesic segment of unit speed. Suppose that (1)* $\mathrm{rad}\left(\frac{d^-\sigma}{dt}(t)\right) > \frac{\pi}{2}$ *and (2)* $\angle_{\sigma(t)}\left(\frac{d^-\sigma}{dt}(t), \frac{d^+\sigma}{dt}(t)\right) = \mathrm{rad}\left(\frac{d^-\sigma}{dt}(t)\right)$ *for $t \in (0, \ell)$. Then $\sigma : [0, \ell] \to M^n$ is called a canonical quasi-geodesic.*

Definition 4.8. *Let M^n be a complete Alexandrov space with curv $\geq k$. Let*

$$W^{reg}(M^n) = \{(p, \vec{w}) | p \in M^n, \vec{w} \in \Sigma_p(M^n), \mathrm{rad}(\vec{w}) > \frac{\pi}{2}\}$$

be the non-extremal portion of unit cones over M^n. The we call $W^{reg}(M^n)$ the canonical regular part of unit cone set over M^n.

We conclude our paper by the following refined version of a problem of Perelman and Petrunin.

Open Problem 4.9. *Let M^n be a complete Alexandrov space with curv $\geq k$. Suppose that M^n has no boundary and $\vec{v} \in W^{reg}(M^n)$ is a unit vector with* $\mathrm{rad}(\vec{v}) > \frac{\pi}{2}$ *in $\Sigma_p(M^n)$. Prove that there exists at most one canonical quasi-geodesic $\sigma : [0, \ell] \to M^n$ with $\sigma'(0) = \vec{v}$.*

In addition, suppose that $U \subset W^{reg}(M^n)$ is a compact measurable subset in regular part and that for each $(p, \vec{v}) \in U$, there is a canonical quasi-geodesic $\varphi_{p,\vec{v}} : [0, \delta] \to M^n$ with

$$\begin{cases} \frac{d^+ \varphi_{p,\vec{v}}}{dt}(0) = \vec{v} \\ \varphi_{p,\vec{v}}(0) = p; \end{cases}$$

and $\psi_t(U) = \{\varphi_{p,\vec{v}}(t) | (p, \vec{v}) \in U\}$. Is it true that

$$\mathrm{Vol}(\psi_t(U)) \leq \mathrm{Vol}(U) \tag{4.10}$$

for $t \geq 0$?

The volume non-increasing property (4.10) might be related to the concavity properties of quasi-geodesics.

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